ISMP 2012

A Heuristic for The Local Region Covering Problem

Ya Ju Fan & Chandrika Kamath

Lawrence Livermore National Laboratory

August 22, 2012

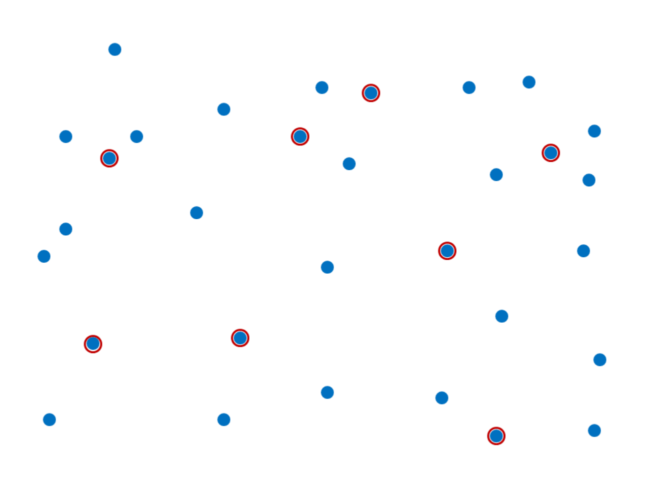
fan4@llnl.gov

LLNL-PRES-564149. This work performed under the auspices of the U.S. Department of Energy by Lawrence Livermore National Laboratory under Contract DE-AC52-07NA27344.

Outline

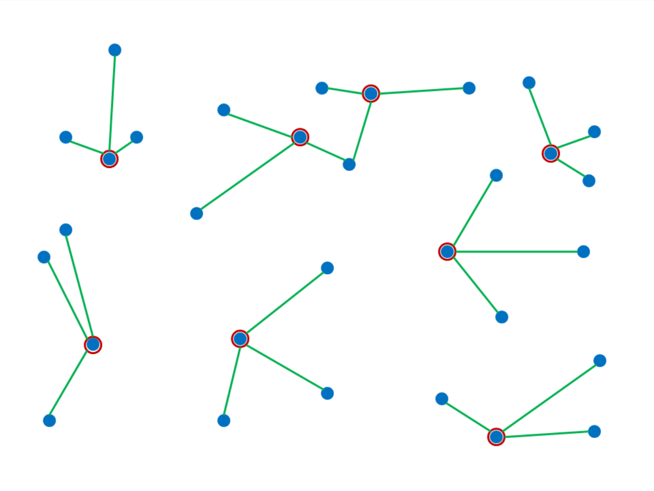
An Example of A Local Region Covering Problem

31 locations and 8 services centers.

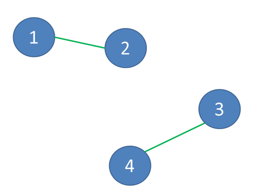


An Example of A Local Region Covering Problem

In this example, $n = 31$, $k = 4$ and $s = 8$.



Example - The Objective Function



$$n = 4$$
$$s = 2$$
$$k = 2$$

$$\begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} \\ D_{21} & D_{22} & D_{23} & D_{24} \\ D_{31} & D_{32} & D_{33} & D_{34} \\ D_{41} & D_{42} & D_{43} & D_{44} \end{bmatrix} \times \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \\ Z_{31} & Z_{32} \\ Z_{41} & Z_{42} \end{bmatrix} \cdot \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \\ T_{31} & T_{32} \\ T_{41} & T_{42} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & D_{12} & D_{13} & D_{14} \\ D_{21} & 0 & D_{23} & D_{24} \\ D_{31} & D_{32} & 0 & D_{34} \\ D_{41} & D_{42} & D_{43} & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 + D_{12} & D_{13} + D14 \\ D_{21} + 0 & D_{23} + D24 \\ D_{31} + D_{32} & 0 + D_{34} \\ D_{41} + D_{42} & D_{43} + 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} D_{12} & 0 \\ 0 & 0 \\ 0 & D_{34} \\ 0 & 0 \end{bmatrix}$$

- D_{ij} is the cost of service from location i to location j.
- s is the total number of services centers (subsets). It should be given.
- $Z_{ip} = 1$ represents assigning location i to p-th service center; 0 otherwise.
- $T_{ip} = 1$ indicating location i is the p-th service center; 0 otherwise.

Local Region Covering

The Optimization Model

$$\min_{Z,T} \sum_{i=1}^{n} \sum_{p=1}^{s} \left(\sum_{j=1}^{n} D_{ij} Z_{jp} \right) \cdot T_{ip}$$

$$\text{s.t.} \quad \sum_{p=1}^{s} Z_{ip} \geqslant 1 \quad \forall i,$$

$$\sum_{i=1}^{n} Z_{ip} = k \quad \forall p,$$

$$\sum_{i=1}^{n} T_{ip} = 1 \quad \forall p,$$

$$\sum_{i=1}^{n} Z_{ip} \cdot T_{ip} = 1 \quad \forall p,$$

$$Z_{ip}, T_{ip} \in \{0, 1\} \quad \forall i, p.$$

- D_{ij} is the cost of service from location i to location j.
- s is the total number of services centers (subsets). It should be given.
- $Z_{ip} = 1$ represents assigning location i to p-th service center; 0 otherwise.
- $T_{ip} = 1$ indicating location i is the p-th service center; 0 otherwise.
- Given $k \cdot s \geqslant n$

Similar Bilinear Programming Formulations: Approach 1

Bilinear Progams

$$\begin{aligned} \max_{x,y} \quad & c^T x + x^T Q^T y + d^T y \\ \text{s.t.} \quad & Ax \leqslant a, \quad B^T y \leqslant b, \\ & x \geqslant 0, \quad y \geqslant 0. \end{aligned}$$

- $A \in \mathcal{R}^{m \times n}$
- $B^T \in \mathcal{R}^{m' \times n'}$
- $Q^T \in \mathcal{R}^{n \times n'}$
- c, d, a, b are vectors of sizes n, n', m, m' respectively.

Equivalent Problem derived from the Duality Theory

$$\begin{aligned} \max_{x,u} \quad & (c^T x + \min \quad b^T u) \\ \text{s.t.} \quad & Ax \leqslant a, \quad Bu \geqslant d + Qx, \\ & x \geqslant 0, \quad u \geqslant 0. \end{aligned}$$

Can't handle our problem!

Having two multiplied variables.

- Gallo and Ulkucu (1977) "Bilinear programming: an exact algorithm". *Mathematical Programming*

Similar Bilinear Programming Formulations: Approach 2

Uncoupled Bilinear Progams

$$\begin{aligned} \max_{x,y,r,s} \quad & xy \\ \text{s.t.} \quad & Cx + Er \geqslant g, \\ & Dy + Fs \geqslant h, \\ & x, y, r, s \geqslant 0. \end{aligned}$$

Coupled Bilinear Progams

$$\begin{aligned} \max_{x,y,r} \quad & xy \\ \text{s.t.} \quad & Cx + Dy + Er \geqslant g, \\ & x, y, r \geqslant 0. \end{aligned}$$

- Vertex Solution
- Convergence and Finite Termination

Our approach!

Solving two subproblems iteratively.

- Bennett and Mangasarian (1993) "Bilinear Separation of Two Sets in n-Space". *Computational Opt. & App.*

Local Region Covering

The Optimization Model

$$\min_{Z,T} \sum_{i=1}^{n} \sum_{p=1}^{s} \left(\sum_{j=1}^{n} D_{ij} Z_{jp} \right) \cdot T_{ip}$$

$$\text{s.t.} \quad \sum_{p=1}^{s} Z_{ip} \geqslant 1 \quad \forall i,$$

$$\sum_{i=1}^{n} Z_{ip} = k \quad \forall p,$$

$$\sum_{i=1}^{n} T_{ip} = 1 \quad \forall p,$$

$$\sum_{i=1}^{n} Z_{ip} \cdot T_{ip} = 1 \quad \forall p,$$

$$Z_{ip}, T_{ip} \in \{0, 1\} \quad \forall i, p.$$

- D_{ij} is the cost of service from location i to location j.
- s is the total number of services centers (subsets). It should be given.
- $Z_{ip} = 1$ represents assigning location i to p-th service center; 0 otherwise.
- $T_{ip} = 1$ indicating location i is the p-th service center; 0 otherwise.
- Given $k \cdot s \geqslant n$

A Bilinear Programming Method

Subproblem 1

Given T, solve

$$\begin{aligned}\min_{Z} \quad & \sum_{i=1}^{n}\sum_{p=1}^{s}\left(\sum_{j=1}^{n} D_{ij}T_{ip}\right)\cdot Z_{jp} \\ \text{s.t.} \quad & \sum_{p=1}^{s} Z_{ip} \geqslant 1 \quad \forall i, \\ & \sum_{i=1}^{n} Z_{ip} = k \quad \forall p, \\ & \sum_{i=1}^{n} Z_{ip}\cdot T_{ip} = 1 \quad \forall p, \\ & Z_{ip} \in \{0,1\} \quad \forall i,p.\end{aligned}$$

Subproblem 2

Given Z, solve

$$\begin{aligned}\min_{T} \quad & \sum_{i=1}^{n}\sum_{p=1}^{s}\left(\sum_{j=1}^{n} D_{ij}Z_{jp}\right)\cdot T_{ip} \\ \text{s.t.} \quad & \sum_{i=1}^{n} T_{ip} = 1 \quad \forall p, \\ & \sum_{i=1}^{n} Z_{ip}\cdot T_{ip} = 1 \quad \forall p, \\ & T_{ip} \in \{0,1\} \quad \forall i,p.\end{aligned}$$

A Bilinear Programming Method - Solving Subproblem 1

Subproblem 1

Given T, solve

$$\min_{Z} \quad \sum_{i=1}^{n}\sum_{p=1}^{s}\left(\sum_{j=1}^{n} D_{ij}T_{ip}\right)\cdot Z_{jp}$$

$$\text{s.t.} \quad \sum_{p=1}^{s} Z_{ip} \geqslant 1 \quad \forall i, \qquad (1)$$

$$\sum_{i=1}^{n} Z_{ip} = k \quad \forall p, \qquad (2)$$

$$\sum_{i=1}^{n} Z_{ip}\cdot T_{ip} = 1 \quad \forall p, \qquad (3)$$

$$Z_{ip} \in \{0,1\} \quad \forall i,p.$$

- The coefficient matrix of Constraint (1) and Constraint (2) is Totally Unimodular.
- k is integer.
- The relaxation problem has integral optimal solution.
- Removing Constraint (3) using the given T.

Totally Unimodular Matrix

$$\sum_{p=1}^{s} Z_{ip} \geqslant 1 \quad \forall i, \qquad (1)$$

$$\sum_{i=1}^{n} Z_{ip} = k \quad \forall p. \qquad (2)$$

Let A be the matrix containing coefficients of the two constraints.

- Elements in A are either 0 or 1 .
- Each column of A contains at most two nonzero coefficients.
- There exists a partition (M_1, M_2) of the set M of rows satisfies $\sum_{i \in M_1} A_{ij} - \sum_{i \in M_2} A_{ij} = 0$ for each column j containing two nonzero coefficients.

Romoving Constraint (3)

Given T,

$$\min_{Z} \sum_{i=1}^{n} \sum_{p=1}^{s} \left(\sum_{j=1}^{n} D_{ij} T_{jp} \right) \cdot Z_{ip}$$

$$\vdots$$

$$\sum_{i=1}^{n} Z_{ip} \cdot T_{ip} = 1 \quad \forall p. \quad (3)$$

Can be replaced by:

$$\min_{Z} \sum_{i=1}^{n} \sum_{p=1}^{s} D_{iC_p} \cdot Z_{ip}$$

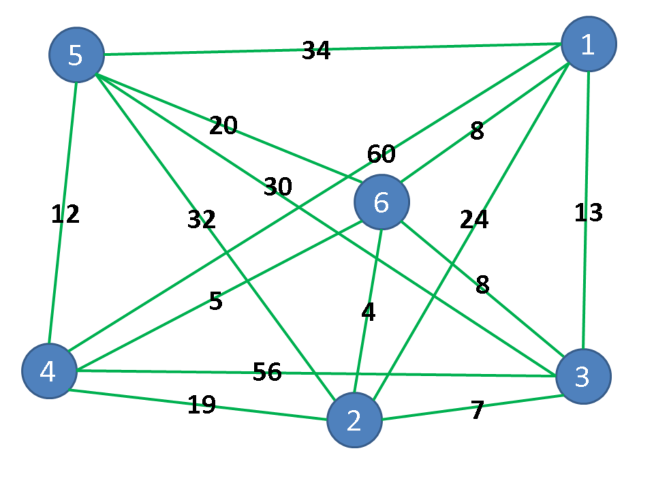
$$\vdots$$

- Let

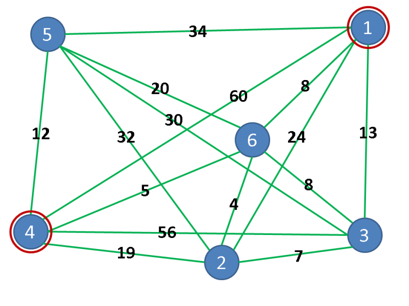
  $$C_p = \{i : T_{ip} = 1 \quad \forall i\} \quad \forall p.$$

- $\sum_{j=1}^{n} D_{ij} T_{jp} = D_{i,C_p}$.
- $Z_{ip} \in \{0, 1\}$.

Example - Local Region Covering



- $n = 6$
- $k = 3$
- $s = 2$
- $k \cdot s \geqslant n$



$$D = \begin{bmatrix} 0 & 24 & 13 & 60 & 34 & 8 \\ 24 & 0 & 7 & 19 & 32 & 4 \\ 13 & 7 & 0 & 56 & 30 & 8 \\ 60 & 19 & 56 & 0 & 12 & 5 \\ 34 & 32 & 30 & 12 & 0 & 20 \\ 8 & 4 & 8 & 5 & 20 & 0 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\overline{D} = D \times T = \begin{bmatrix} 0 & 24 & 13 & 60 & 34 & 8 \\ 24 & 0 & 7 & 19 & 32 & 4 \\ 13 & 7 & 0 & 56 & 30 & 8 \\ 60 & 19 & 56 & 0 & 12 & 5 \\ 34 & 32 & 30 & 12 & 0 & 20 \\ 8 & 4 & 8 & 5 & 20 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 60 \\ 24 & 19 \\ 13 & 56 \\ 60 & 0 \\ 34 & 12 \\ 8 & 5 \end{bmatrix}$$

$$\min_{Z} \quad \sum_{i=1}^{n} \sum_{p=1}^{s} \left(\sum_{j=1}^{n} \overline{D}_{ip} \right) \cdot Z_{jp}$$

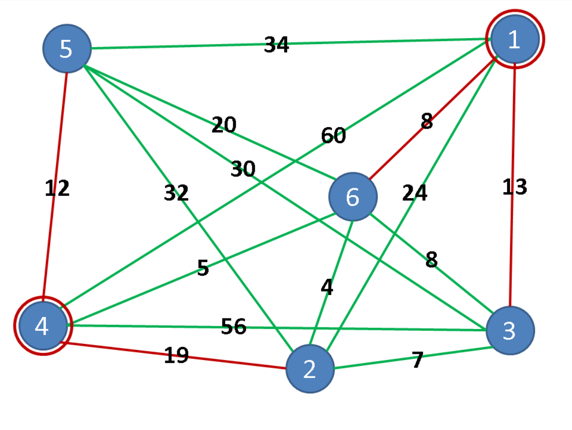
$$\text{s.t.} \quad \sum_{p=1}^{s} Z_{ip} \geqslant 1 \quad \forall i,$$

$$\sum_{i=1}^{n} Z_{ip} = k \quad \forall p,$$

~~$$\sum_{i=1}^{n} Z_{ip} \cdot T_{ip} = 1 \quad \forall p,$$~~

$$0 \leqslant Z_{ip} \leqslant 1 \quad \forall i, p.$$

Solution



$$\text{Obj} = 52$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

A Bilinear Programming Method - Solving Subproblem 2

Subproblem 2

Given Z, solve

$$\min_{T} \quad \sum_{i=1}^{n} \sum_{p=1}^{s} \left(\sum_{j=1}^{n} D_{ij} Z_{jp} \right) \cdot T_{ip}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} T_{ip} = 1 \quad \forall p, \qquad (4)$$

$$\sum_{i=1}^{n} Z_{ip} \cdot T_{ip} = 1 \quad \forall p, \qquad (5)$$

$$T_{ip} \in \{0, 1\} \quad \forall i, p.$$

- Removing Constraint (5) using the given Z.
- Selecting the minimum sum $\hat{D}_{i^*p}$ (will be defined) for each p.

Removing Constraint (5)

Given T,

$$\min_{Z} \sum_{i=1}^{n}\sum_{p=1}^{s}\left(\sum_{j=1}^{n} D_{ij}Z_{jp}\right)\cdot T_{ip}$$

$$\sum_{i=1}^{n} T_{ip} = 1 \quad \forall p,$$

$$\sum_{i=1}^{n} Z_{ip}\cdot T_{ip} = 1 \quad \forall p. \quad (5)$$

Can be replaced by:

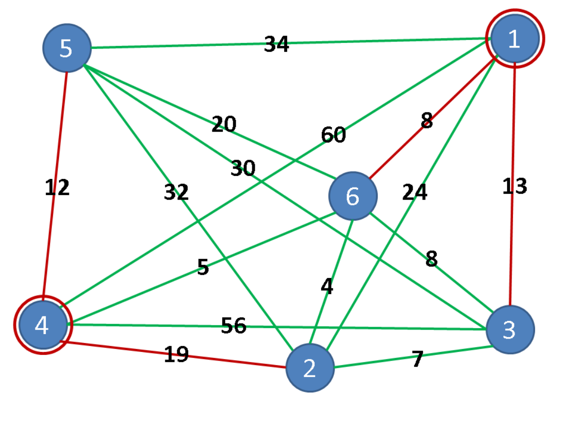
$$\min_{Z} \sum_{p=1}^{s}\sum_{i\in E_p}\left(\sum_{j\in E_p} D_{ij}\right)\cdot T_{ip}$$

$$\sum_{i=1}^{n} T_{ip} = 1 \quad \forall p.$$

- Let $E_p = \{i : Z_{ip} = 1 \quad \forall i\} \quad \forall p.$
- $\sum_{j=1}^{n} D_{ij}Z_{jp} = \sum_{j\in E_p} D_{ij}$
- $T_{ip} \in \{0, 1\}$.
- Selecting the minimum sum for each p:

$$\hat{D}_{i^*p} = \min_{i\in E_p} \sum_{j\in E_p} D_{ij}.$$

Example



$$\text{Obj} = 52$$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

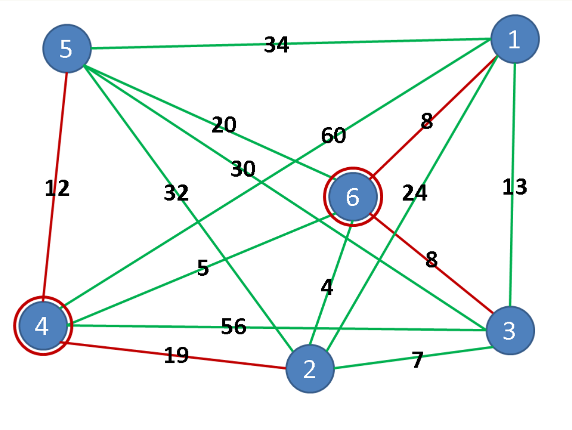
Example

$$D \times Z = \begin{bmatrix} 0 & 24 & 13 & 60 & 34 & 8 \\ 24 & 0 & 7 & 19 & 32 & 4 \\ 13 & 7 & 0 & 56 & 30 & 8 \\ 60 & 19 & 56 & 0 & 12 & 5 \\ 34 & 32 & 30 & 12 & 0 & 20 \\ 8 & 4 & 8 & 5 & 20 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0+13+8 & 24+60+34 \\ 24+7+4 & 0+19+32 \\ 13+0+8 & 7+56+30 \\ 60+56+5 & 19+0+12 \\ 34+30+20 & 32+12+0 \\ 8+8+0 & 4+5+20 \end{bmatrix}$$

$$\min_{Z} \quad \sum_{p=1}^{s} \sum_{i \in E_p} \left(\sum_{j \in E_p} D_{ij} \right) \cdot T_{ip}$$
$$\sum_{i=1}^{n} T_{ip} = 1 \quad \forall p.$$

The Optimal Objective Value
$= \min(21,21,16) + \min(51,31,44) = 16 + 31 = 47 < 52.$

Solution



$$\text{Obj} = 47$$

$$T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

The Algorithm

Subproblem 1

Given T, solve the LP:

$$\begin{aligned}
\min_{Z} \quad & \sum_{i=1}^{n} \sum_{p=1}^{s} \left(\sum_{j=1}^{n} D_{ij} T_{jp} \right) \cdot Z_{jp} \\
\text{s.t.} \quad & \sum_{p=1}^{s} Z_{ip} \geqslant 1 \quad \forall i, \\
& \sum_{i=1}^{n} Z_{ip} = k \quad \forall p, \\
& 0 \leqslant Z_{ip} \leqslant 1 \quad \forall i, p.
\end{aligned}$$

Subproblem 2

Given Z, selecting the minimum sum for each p:

$$\hat{D}_{i^* p} = \min_{i \in E_p} \sum_{j \in E_p} D_{ij}.$$

The optimal T has elements

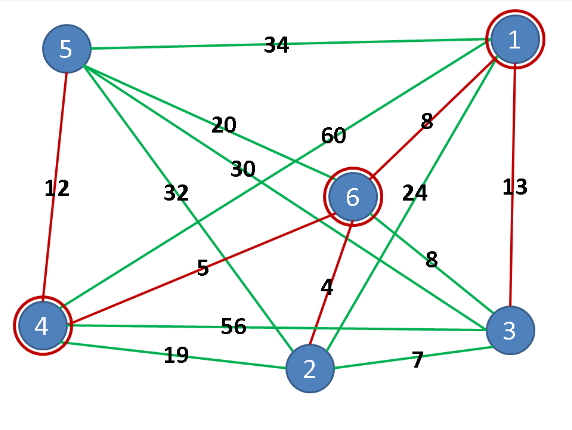
$$T_{lp} = \begin{cases} 1 & \text{if } l = \arg \min_{i \in E_p} \sum_{j \in E_p} D_{ij}; \\ 0 & \text{otherwise.} \end{cases}$$

Iterate untill converge!

Conclusion

- A bilinear programming problem
- Two multiplied variables
    - Objective function
    - A constraint
- Bilinear programming algorithm
    - Subproblem 1 has integral LP solutions.
    - Subproblem 2 is a simple minimum value selection problem.
- An iterative method with finite termination
- Simple and easy!

Extension



$$
\begin{aligned}
\text{Obj} &= 9 + 17 + 21 \\
&= 47
\end{aligned}
$$

$$
T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}
$$